Mechanics Research Communications 38 (2011) 29–33

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

A general solution of the axisymmetric contact problem for biphasic cartilage layers

Ivan Argatov

Institute of Mathematics and Physics, Aberystwyth University, Ceredigion SY23 3BZ, Wales, UK

ARTICLE INFO

Article history:
Received 14 July 2010
Received in revised form
18 November 2010
Available online 24 December 2010

Keywords:
Contact problem
Cartilage layer
Biphasic material model
Analytical solution

ABSTRACT

A unilateral axisymmetric contact problem for articular cartilage layers is considered. The articular cartilages bonded to subchondral bones are modeled as biphasic materials consisting of a solid phase and a fluid phase. It is assumed that the subchondral bones are rigid and shaped like bodies of revolution with arbitrary convex profiles. The obtained closed-form analytical solution is valid over time periods compared with the typical diffusion time and can be used for increasing loading.

1. Introduction

Biomechanical contact problems involving transmission of forces across biological joints are of considerable practical importance in orthopedic surgery and many numerical models for contact interaction of articular cartilage surfaces in joints are available (Wilson et al., 2005; Han et al., 2005; Anderson et al., 2008). At the same time, the necessity of analytical models becomes an important issue in developing improved understanding of load distribution in the normal and pathological joints, which affects the mechanical aspects of osteoarthritis (Wu et al., 2000). It is known that the joint degradation in the early stages of osteoarthritis may be reflected in changes of material properties of articular cartilage layers, which were observed to become thicker, softer, and more permeable (see references given by Wu et al. (2000)). Thus, having at hand an analytical model of the joint, it is easy to predict the corresponding behavior of the important contact parameters such as the maximum contact pressure and the contact area during the evolution of osteoarthritis in its early stages. Modeling of articular cartilage replacement materials also requires an analytical description of articular contact mechanics (Stoffel et al., 2009). In particular, analytical models would be useful in studying the structural optimization problem for synthetic implants for the local repair of full-thickness cartilage defects (Messner and Gillquist, 1993). On the other hand, analytical solutions are used to test the accuracy of numerical models (Wu et al., 1997a).

As it was observed by Genda et al. (2001), a practical and easy-to-use analysis technique for studying the patient's hip joint contact pressure distribution would be useful to assess the effect of abnormal biomechanical conditions in the hip joint where the role of an ideal sphericity in normal function of hip joint is very important. It should be noted that the method of Genda et al. (2001) is based on the discrete element analysis and the contact interaction between the articular joint surfaces is modeled through the interaction of a series of normal and shear springs.

There is a large body of literature associated with contact interaction of thin layers (Eberhardt et al., 1991; Barry and Holmes, 2001; Hlaváček, 2008). Even in the case of pure elastic behavior of the material, the contact problem presents significant difficulties for analytical solution (Argatov, 2005). But the contact problem for biphasic layers is time-dependent, and such problems have not been widely investigated before. Ateshian et al. (1994) obtained an asymptotic solution for the axisymmetric contact problem of two identical biphasic cartilage layers attached to two rigid impermeable spherical bones of equal radii modeled as circular paraboloids. Wu et al. (1996) extended this solution by combining the joint contact model for the contact of two biphasic cartilages with the assumption of the kinetic relationship from classical contact mechanics (Johnson, 1985).

The biphasic cartilage constitutive model proposed by Mow et al. (1980) has proved successful in describing the mechanical response of articular cartilage. That is why, the asymptotic model developed by Ateshian et al. (1994) and Wu et al. (1996) has received much attention in the recent years. In particular, Wu et al. (1997b) obtained an improved solution for the contact of two biphasic cartilage layers which can be used for dynamic loading.

E-mail address: ivan.argatov@gmail.com

0093-6413/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mechrescom.2010.11.006

Mishuris and Argatov (2009) extended the analysis of Wu et al. (1996) by formulating the refined contact condition which takes into account the tangential displacements at the contact region. A simplified analytical solution for the radial and tangential displacements on the surface of a hemispherical layer of porous-elastic articular cartilage was recently obtained by Quiñonez et al., 2010.

However, it should be emphasized that the analytical solutions obtained by Ateshian et al. (1994) and Wu et al. (1996) are restricted to the circular paraboloid geometry of the contacting surfaces. In this study, the method of Argatov (2004) is adopted to solve the general axisymmetric contact problem for biphasic cartilage layers with the arbitrary joint geometry resulting in the circular contact area.

2. Formulation of the contact problem

We consider an axisymmetric contact between two thin linear biphasic cartilage layers firmly attached to rigid bones shaped like bodies of revolution. Introducing the cylindrical coordinate system (r, φ, z), we write the equations of the cartilage surfaces (before loading) in the form $z=(-1)^n\Phi_n(r)$ $(n=1, 2)$. In the particular case of bones shaped like paraboloids of revolution, we have $\Phi_n(r)=(2R_n)^{-1}r^2$ $(n=1, 2)$, where R_1 and R_2 are the curvature radii of the bone surfaces at their apexes.

Denoting the vertical approach of the bones by $-\delta_0(t)$, we write the linearized unilateral contact non-penetration condition in the form

$$\delta_0(t)-(w_1(r,t)+w_2(r,t)) \le \Phi_1(r)+\Phi_2(r). \tag{1}$$

According to Ateshian et al. (1994), the vertical displacement of the boundary points of the tissue, $w_n(r, t)$, is expressed through is the contact pressure $P(r, t)$ as follows:

$$\begin{aligned} w_n(r,t) = {}& \mu_{sn}^{-1}h_n^3\{(3r)^{-1}\partial_r(r\partial_r P(r,t)) \\ & +h_n^{-2}\mu_{sn}k_n\int_0^t r^{-1}\partial_r(r\partial_r P(r,\tau))\,\mathrm{d}\tau\}. \end{aligned} \tag{2}$$

Here, μ_{sn} is the shear modulus of the solid phase of the cartilage tissue $(n=1, 2)$, h_1 and h_2 are the thicknesses of the cartilage layers, k_1 and k_2 are the cartilage permeabilities.

The equality in relation (1) determines the contact radius $a(t)$. In other words, the following equation holds within the contact area:

$$w_1(r,t)+w_2(r,t)=\delta_0(t)-\Phi(r), \quad r \le a(t) \tag{3}$$

with $\Phi(r)=\Phi_1(r)+\Phi_2(r)$. Thus, we have

$$\Phi(r)=\frac{R_1+R_2}{2R_1R_2}r^2. \tag{4}$$

Substituting the expressions for the displacements $w_1(r, t)$ and $w_2(r, t)$ given by formula (2) into Eq. (3), we write the contact condition in the form

$$r^{-1}\partial_r(r\partial_r P(r,t))+\chi\int_0^t r^{-1}\partial_r(r\partial_r P(r,\tau))\,\mathrm{d}\tau = m(\Phi(r)-\delta_0(t)), \tag{5}$$

where $\chi = h_1^{-2}3\mu_{s1}k_1 + h_2^{-2}3\mu_{s2}k_2$ and $m = ((3\mu_{s1})^{-1}h_1^3+(3\mu_{s2})^{-1}h_2^3)^{-1}$. Eq. (5) is used to find the contact pressure $P(r, t)$. The contact radius $a(t)$ is determined from the condition that the contact pressure vanishes at the contour of the contact area:

$$P(r,t)\ge 0, \quad r\le a(t); \qquad P(a(t),t)=0. \tag{6}$$

Moreover, in the case of contact problems for biphasic cartilage layer, in which the contact pressure is carried primarily by the fluid phase, additionally a smooth transition of the surface normal stresses is assumed from the contact region $r<a(t)$ to the outside region $r>a(t)$. From physical point of view, the contact pressure exerted by an axisymmetric blunt punch should satisfy the regularity condition at the punch apex. Thus, the following boundary conditions are imposed (Ateshian et al., 1994):

$$\partial_r P(r,t)|_{r=a(t)}=0, \qquad \partial_r P(r,t)|_{r=0}=0. \tag{7}$$

The equilibrium equation for the whole system is

$$2\pi\int_0^{a(t)} P(\rho,t)\rho\,\mathrm{d}\rho = F(t), \tag{8}$$

where $F(t)$ denotes the external load. For non-decreasing loads when $dF(t)/\mathrm{d}t \ge 0$, the contact radius increases monotonously, i. e., $\mathrm{d}a(t)/\mathrm{d}t>0$ (Wu et al., 1997b).

The aim of this study is to derive a general solution for the axisymmetric contact problem for biphasic cartilage layers formulated by Eq. (5) in the general case of arbitrary joint geometry under the assumption of increasing loading.

3. Equation for the displacement parameter

Integrating Eq. (5) with respect to r, we get

$$\partial_r P(r,t)+\chi\int_0^t \partial_r P(r,\tau)\,\mathrm{d}\tau = m\left(r^{-1}\int_0^r \Phi(\rho)\rho\,\mathrm{d}\rho-\delta_0(t)\frac{r}{2}\right). \tag{9}$$

Note that the constant of integration vanishes due to the regularity condition.

After substituting the value $r=a(t)$ and taking into account the boundary conditions (6) and (7), Eq. (9) transforms into the equation

$$\delta_0(t)=\frac{2}{a^2(t)}\int_0^{a(t)}\Phi(\rho)\rho\,\mathrm{d}\rho. \tag{10}$$

Eq. (10) connects the unknown contact radius $a(t)$ and punch displacement $\delta_0(t)$.

4. Equation for the radius of the contact area

After integration and changing the order of integration, Eq. (9) takes the form

$$\begin{aligned} & P(r,t)+\chi\int_0^t P(r,\tau)\,\mathrm{d}\tau \\ & = m\left(\int_0^r \Phi(\rho)\rho\ln\frac{r}{\rho}\,\mathrm{d}\rho-\delta_0(t)\frac{r^2}{4}\right)+D_2(t). \end{aligned} \tag{11}$$

By using the boundary condition (6), the constant of integration can be obtained as a function of the contact radius $a(t)$:

$$D_2(t)=\frac{m}{4}\delta_0(t)a^2(t)-m\int_0^{a(t)}\Phi(\rho)\rho\ln\frac{a(t)}{\rho}\,\mathrm{d}\rho. \tag{12}$$

Further, we multiply both sides of Eq. (11) by r and integrate over the contact radius. After changing the order of integration and taking account of (8) and (12), we obtain

$$\begin{aligned} & F(t)+\chi\int_0^t F(\tau)\,\mathrm{d}\tau \\ & = \pi m\left\{\delta_0(t)\frac{a^4(t)}{8}-\frac{1}{2}\int_0^{a(t)}\Phi(\rho)\rho(a^2(t)-\rho^2)\mathrm{d}\rho\right\}. \end{aligned} \tag{13}$$

Finally, in view of Eq. (10), Eq. (13) takes the form

$$\frac{\pi m}{4}\int_0^{a(t)}\Phi(\rho)\rho(2\rho^2-a^2(t))\mathrm{d}\rho = F(t)+\chi\int_0^t F(\tau)\,\mathrm{d}\tau. \tag{14}$$

Eq. (14) connects the unknown radius of the contact area $a(t)$ and the known contact load $F(t)$. Note that in the case (4), Eqs. (10) and (14) coincide with the corresponding result obtained by Wu et al. (1997b) in a different way.

5. Example. General paraboloid of revolution

Now let us assume that the gap between the cartilage layers is described by

$$\Phi(r) = Cr^{\lambda}, \tag{15}$$

where C and λ are constants. In this case, Eqs. (10) and (14) take the form

$$\delta_0(t) = \frac{2C}{\lambda+2}\, a^{\lambda}(t). \tag{16}$$

$$\frac{\pi m\lambda C a^{\lambda+4}(t)}{4(\lambda+2)(\lambda+4)} = F(t) + \chi\int_0^t F(\tau)\,\mathrm{d}\tau. \tag{17}$$

It is readily seen that Eq. (17) allows to determine the contact radius $a(t)$ as

$$a(t) = \left(\frac{4(\lambda+2)(\lambda+4)}{\pi m\lambda C}\right)^{1/(\lambda+4)} \left(F(t) + \chi\int_0^t F(\tau)\,\mathrm{d}\tau\right)^{1/(\lambda+4)}. \tag{18}$$

Substituting now the obtained solution (18) into Eq. (16), we derive the following formula for the approach displacement $\delta_0(t)$:

$$\delta_0(t) = \frac{2C}{\lambda+2}\left(\frac{4(\lambda+2)(\lambda+4)}{\pi m\lambda C}\right)^{\lambda/(\lambda+4)} \times\left(F(t) + \chi\int_0^t F(\tau)\,\mathrm{d}\tau\right)^{\lambda/(\lambda+4)}. \tag{19}$$

Since we consider monotonic loading (the function $F(t)$ does not decrease with time), the utilized mathematical model of contact shows (see, in particular, Eq. (18)) that $a(t)$ will increase up to infinity. Assuming reasonable growth of the function $F(t)$ that is $F(t)\to F_\infty$ as $t\to\infty$, the following asymptotic estimate holds true:

$$a(t) \sim \left(\frac{4(\lambda+2)(\lambda+4)\chi}{\pi m\lambda C} F_\infty t\right)^{1/(\lambda+4)}, \quad t\to\infty. \tag{20}$$

As a result derived from Eq. (19), we obtain the following asymptotic expansion:

$$\delta_0(t) \sim \frac{2C}{\lambda+2}\left(\frac{4(\lambda+2)(\lambda+4)\chi}{\pi m\lambda C} F_\infty t\right)^{\lambda/(\lambda+4)}, \quad t\to\infty, \tag{21}$$

Note that in the case (4), formulas (20) and (21) coincide with the corresponding result obtained by Wu et al. (1997b) in a different way.

Note also that as it was observed by Barry and Holmes (2001), Ateshian et al. (1994) considered the small time scales, and Eq. (2) is not valid for long times compared with the typical diffusion time $\tau_n' = h_n^2/(k_n H_{An})$, where H_{An} is the aggregate modulus of the n-th cartilage layer given by $H_{An} = \lambda_{sn} + 2\mu_{sn}$ with λ_s and μ_s being the Lamé coefficients of cartilage. As a consequence of this, the asymptotic model (18)–(21) cannot describe the behavior of articular cartilage near the compaction point, i.e., the point where all pore fluid is squeezed out and the load is fully carried by the extracellular matrix.

6. Contact pressure

Taking into account formula (12), we rewrite Eq. (11) as follows:

$$P(r,t) + \chi\int_0^t P(r,\tau)\,\mathrm{d}\tau = \frac{m}{4}\,\delta_0(t)(a^2(t)-r^2) + m\left(\int_0^r \Phi(\rho)\rho\ln\frac{r}{\rho}\,\mathrm{d}\rho - \int_0^{a(t)} \Phi(\rho)\rho\ln\frac{a(t)}{\rho}\,\mathrm{d}\rho\right). \tag{22}$$

Let us first consider the special case (15) when Eq. (22) takes the form

$$P(r,t) + \chi\int_0^t P(r,\tau)\,\mathrm{d}\tau = \frac{m}{4}\,\delta_0(t)(a^2(t)-r^2) - \frac{mC}{(\lambda+2)^2}\left(a^{\lambda+2}(t) - r^{\lambda+2}\right). \tag{23}$$

Now we introduce the following notation for the operator on the left-hand side of Eq. (23):

$$\mathcal{K}y(t) = y(t) + \chi\int_0^t y(\tau)\,\mathrm{d}\tau. \tag{24}$$

The inverse operator to $\mathcal{K}$ denoted by $\mathcal{K}^{-1}$ is defined by the formula

$$\mathcal{K}^{-1}Y(t) = Y(t) - \chi\int_0^t Y(\tau)e^{-\chi(t-\tau)}\mathrm{d}\tau. \tag{25}$$

In view of (16), (24) and (25), a solution of Eq. (23) can be represented as follows:

$$P(r,t) = \frac{mC}{2(\lambda+2)^2}\,\mathcal{K}^{-1}\left(\lambda a^{\lambda+2}(t) - (\lambda+2)a^{\lambda}(t)r^2 + 2r^{\lambda+2}\right). \tag{26}$$

Finally, let $H(x)$ be the Heaviside step function defined as $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x \le 0$. Then, taking the notation (25) into account, we rewrite formula (26) in the form

$$\frac{2(\lambda+2)^2}{mC}P(r,t) = \lambda a^{\lambda+2}(t) - (\lambda+2)a^{\lambda}(t)r^2 + 2r^{\lambda+2} - \chi\int_0^t e^{-\chi(t-\tau)}\left(\lambda a^{\lambda+2}(\tau) - (\lambda+2)a^{\lambda}(\tau)r^2 + 2r^{\lambda+2}\right) \times H(a(\tau)-r)\,\mathrm{d}\tau. \tag{27}$$

In the general case, in view of (24) and (25), a solution of Eq. (22) can be represented as

$$P(r,t) = \frac{m}{4}\,\mathcal{K}^{-1}(\delta_0(t)(a^2(t)-r^2)) + m\mathcal{K}^{-1}\left(\int_0^r \Phi(\rho)\rho\ln\frac{r}{\rho}\,\mathrm{d}\rho - \int_0^{a(t)} \Phi(\rho)\rho\ln\frac{a(t)}{\rho}\,\mathrm{d}\rho\right). \tag{28}$$

Formula (28) is the sought-for general solution of Eq. (22).

7. Example. Effect of deviation from sphericity

If two spherical cartilage layers contact each other as shown in Fig. 1, then Eq. (4) gives the first order approximation for the initial gap between the surfaces of contacting layers. Representing the spherical surfaces in the cylindrical coordinate system by the equations $(z_2 - R_2)^2 + r^2 = R_2^2$ and $(z_1 + R_1)^2 + r^2 = R_1^2$ with $R_1 < 0$, we determine the initial gap as $z_2 - z_1 = \Phi(r)$. Expanding the function $z_2 - z_1$ in a power series with respect to r and neglecting all terms of order r^6 or higher, we obtain the second order approximation

$$\Phi(r) = \frac{R_1+R_2}{2R_1R_2}r^2 + \frac{R_1^3+R_2^3}{8R_1^3R_2^3}r^4. \tag{29}$$

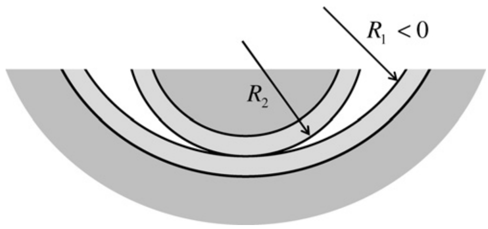

Fig. 1. Schematic diagram of the axisymmetric articular cartilage contact problem.

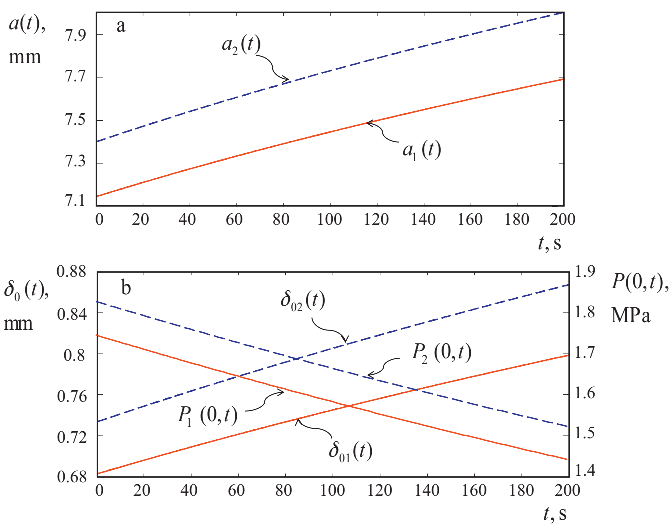

Fig. 2. Comparison of the contact radius (a), indentation parameter and maximum contact pressure (b).

In order to illustrate the obtained analytical solution in the case (29), we adopt the following typical cartilage material parameters: $\mu_s = 0.25$ MPa, $k = 2 \times 10^{-3}$ mm^4 N^{-1} s^{-1}, $h = 1$ mm (Ateshian et al., 1994). Also, we adopt the example of joint geometry with the radii of curvature $R_1 = -20$ mm and $R_2 = 10$ mm for the patella and patellar groove, respectively, considered in Herzog et al. (1998). Finally, a constant contact load $F(t) = 100$ N is applied to the joint in the quasi-static progressive loading test.

In Fig. 2, we present a comparison of the results obtained for the radius of the contact zone, $a(t)$, the displacement parameter, $\delta_0(t)$, and the maximum contact pressure, $P(0, t)$. Fig. 3 shows the contact pressures for three stages of deformation corresponding to the moments of time $t = 0$, 100 s, and 200 s. It is clear from the obtained numerical results that the difference between the solutions corresponding to the two approximations (4) and (29) slightly increases with time. For the contact radius, $a(t)$, the maximum contact pressure (at the center of the contact area), $P(0, t)$, and the displacement parameter, $\delta_0(t)$, the maximum errors are 2.3%, 3.3%, and 4.6% for $t = 200$ s, respectively. Thus, the displacement parameter is the most sensitive to the accuracy of approximating the joint geometry.

Observe also that the replacement of approximation (4) by the approximation (29), which describes the spherical shapes of the contacting cartilage layers more precisely, results in a decrease of the contact zone and in an increase of the maximum contact pressure and the approach of the contacting cartilage layers.

8. Discussion and conclusion

In the example considered above, the maximum relative difference between the solutions was approximately 5%. It should be emphasized that this is a systematic error because it is originated from the geometrical representation of the contact surfaces. At the same time, the difference between the solutions can become more pronounced (5% in the indentation depth) for other reasonable sets of the problem parameters.

Wu et al. (1997a) compared the solution obtained using ABAQUS software with the analytical solution of Wu et al. (1996) in the analogous axisymmetric joint contact test. The maximum relative difference between these two solutions was observed as much as 10%. This implies that the additional systematic error associated with the joint geometry representation is not admissible. Note also that the difference between the analytical approximate solution based on idealized joint geometry and an exact solution using subject-specific (i.e., irregular physiological) geometry can be estimated by application of finite element models (Anderson et al., 2010).

The accuracy of mathematical models become even more important in the elaboration of experimental results. Herzog et al. (1998) performed an experimental study to quantify the in situ joint contact mechanics. In particular, the total contact area and peak pressure in the patellofemoral joint of the cat were obtained in situ using Fuji Pressensor film. The experimental results are in conceptual agreement with the theoretical predictions due to Wu et al. (1996). At that, it should be kept in mind that Fuji pressure sensitive film has an estimated accuracy of ±10% (Singerman et al., 1987).

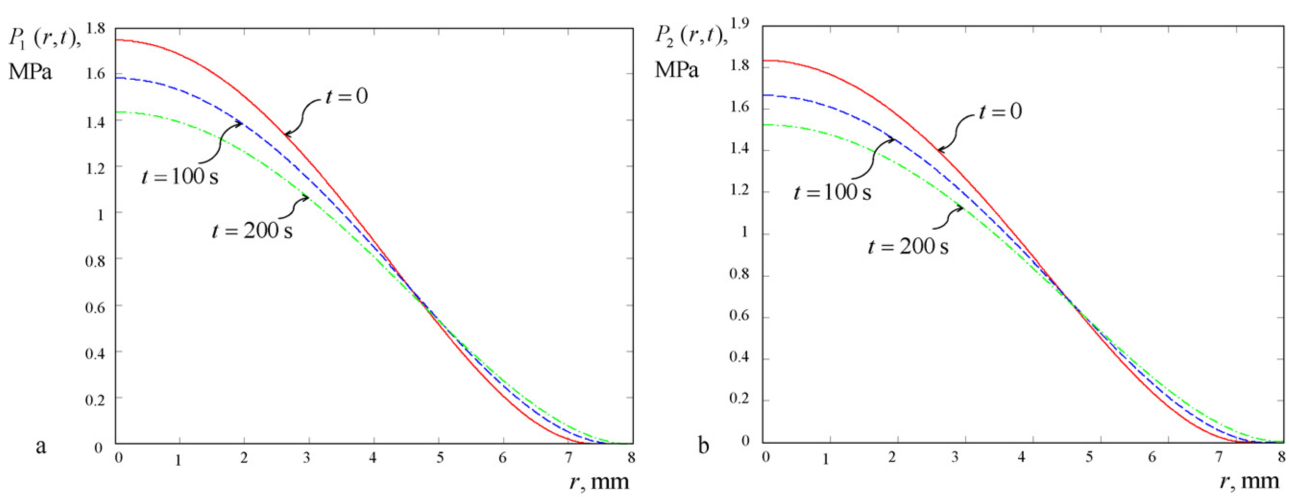

Fig. 3. Contact pressure in the case of the first order (a) and the second oder (b) approximations.

It should be emphasized that we present a general solution of the axisymmetric contact problem for biphasic cartilage layers in the framework of the biomechanical model developed by Ateshian et al. (1994) and Wu et al. (1996). The field of application of the described method is restricted to specific situations when the criteria for using the biomechanical model are satisfied, including the requirements on the joint geometry, mechanical properties, and loading conditions.

In the present study, the general solution of the axisymmetric contact problem for biphasic cartilage layers has been obtained. Closed-form formulas (10), (14) and (28) for evaluating the displacement parameter, $\delta_0(t)$, contact radius, $a(t)$, and contact pressure, $P(r, t)$, constitute the main results of the present study. In the special case (4), when the contacting cartilage layers are modelled as elliptic paraboloids, the obtained Eqs. (10) and (14) coincide with the results obtained by Wu et al. (1997b). The new closed-form solution for the contact pressure (28) derived here allows to carry out a complete parametric analysis of the articular contact mechanics taking into account the effect of the axisymmetric joint geometry.

Acknowledgments

The author gratefully acknowledges the hospitality of the Institute of General Mechanics, RWTH Aachen University, where this work was carried out during his stay at the IAM in October 2009. The author is indebted to Professor D. Weichert and Dr. M. Stoffel for support of this work and valuable discussion. The author also would like to express his gratitude to the Referees for their helpful comments and discussions.

References

Anderson, A.E., Ellis, B.J., Maas, S.A., Peters, C.L., Weiss, J.A., 2008. Validation of finite element predictions of cartilage contact pressure in the human hip joint. J. Biomech. Eng. 130 (5), 051008.

Anderson, A.E., Ellis, B.J., Maas, S.A., Weiss, J.A., 2010. Effects of idealized joint geometry on finite element predictions of cartilage contact stresses in the hip. J. Biomech. 43, 1351–1357.

Argatov, I.I., 2004. Approximate solution of an axisymmetric contact problem with allowance for tangential displacements on the contact surface. J. Appl. Mech. Tech. Phys. 45, 118–123.

Argatov, I.I., 2005. The pressure of a punch in the form of an elliptic paraboloid on a thin elastic layer. Acta Mech. 180, 221–232.

Ateshian, G.A., Lai, W.M., Zhu, W.B., Mow, V.C., 1994. An asymptotic solution for the contact of two biphasic cartilage layers. J. Biomech. 27, 1347–1360.

Barry, S.I., Holmes, M., 2001. Asymptotic behaviour of thin poroelastic layers. IMA J. Appl. Math. 66, 175–194.

Eberhardt, A.W., Lewis, J.L., Keer, L.M., 1991. Normal contact of elastic spheres with two elastic layers as a model of joint articulation. J. Biomech. Eng. 113, 410–417.

Genda, E., Iwasaki, N., Li, G., MacWilliams, B.A., Barrance, P.J., Chao, E.Y.S., 2001. Normal hip joint contact pressure distribution in single-leg standing-effect of gender and anatomic parameters. J. Biomech. 34, 895–905.

Han, S.K., Federico, S., Epstein, M., Herzog, W., 2005. An articular cartilage contact model based on real surface geometry. J. Biomech. 38, 179–184.

Herzog, W., Diet, S., Suter, E., Mayzus, P., Leonard, T.R., Müller, C., Wu, J.Z., Epstein, M., 1998. Material and functional properties of articular cartilage and patellofemoral contact mechanics in an experimental model of osteoarthritis. J. Biomech. 31, 1137–1145.

Hlaváček, M., 2008. Elliptical contact on elastic incompressible coatings. Engineering Mechanics 15, 249–261.

Johnson, K.L., 1985. Contact Mechanics. Cambridge University Press, Cambridge, UK.

Messner, K., Gillquist, J., 1993. Synthetic implants for the repair of osteochondral defects of the medial femoral condyle: A biomechanical and histological evaluation in the rabbit knee. Biomaterials 14, 513–521.

Mishuris, G., Argatov, I., 2009. Exact solution to a refined contact problem for biphasic cartilage layers. In: Swansea, U.K., Nithiarasu, P. Löhner, R. (Eds.), Proc. 1st International Conference on Mathematical and Computational Biomedical Engineering—CMBE2009, June 29–July 1, 2009. 151–154.

Mow, V.C., Kuei, S.C., Lai, W.M., Armstrong, C.G., 1980. Biphasic creep and stress relaxation of articular cartilage in compression: theory and experiments. J. Biomech. Eng. 102, 73–84.

Quiñonez, A.F., Summers, J.L., Fisher, J., Jin, Z.M., 2010. An analytical solution for the radial and tangential displacements on a thin hemispherical layer of articular cartilage. Biomech. Model. Mechanobiol. doi:10.1007/s10237-010-0234-6.

Singerman, R.J., Pedersen, D.R., Brown, T.D., 1987. Quantitation of pressure-sensitive film using digital image scanning. Exp. Mech. 27, 99–105.

Stoffel, M., Weichert, D., Müller-Rath, R., 2009. Modeling of articular cartilage replacement materials. Arch. Mech. 61, 69–87.

Wilson, W., van Donkelaar, C.C., van Rietbergen, R., Huiskes, R., 2005. The role of computational models in the search for the mechanical behavior and damage mechanisms of articular cartilage. Med. Eng. Phys. 27, 810–826.

Wu, J.Z., Herzog, W., Ronsky, J., 1996. Modeling axi-symmetrical joint contact with biphasic cartilage layers—an asymptotic solution. J. Biomech. 29, 1263–1281.

Wu, J.Z., Herzog, W., Epstein, M., 1997a. Evaluation of the finite element software ABAQUS for biomechanical modelling of biphasic tissues. J. Biomech. 31, 165–169.

Wu, J.Z., Herzog, W., Epstein, M., 1997b. An improved solution for the contact of two biphasic cartilage layers. J. Biomech. 30, 371–375.

Wu, J.Z., Herzog, W., Epstein, M., 2000. Joint contact mechanics in the early stages of osteoarthritis. Med. Eng. Phys. 22, 1–12.